

BOUNDARY-LAYERS FOR A NEUMANN PROBLEM AT HIGHER CRITICAL EXPONENTS

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ABSTRACT. We consider the Neumann problem

$$(P) \quad -\Delta v + v = v^{q-1} \text{ in } \mathcal{D}, \quad v > 0 \text{ in } \mathcal{D}, \quad \partial_\nu v = 0 \text{ on } \partial\mathcal{D},$$

where \mathcal{D} is an open bounded domain in \mathbb{R}^N , ν is the unit inner normal at the boundary and $q > 2$. For any integer, $1 \leq h \leq N - 3$, we show that, in some suitable domains \mathcal{D} , problem (P) has a solution which blows-up along a h -dimensional minimal submanifold of the boundary $\partial\mathcal{D}$ as q approaches from either below or above the higher critical Sobolev exponent $\frac{2(N-h)}{N-h-2}$.

1. INTRODUCTION

We are interested in the classical Neumann problem

$$(1) \quad -\varepsilon^2 \Delta v + v = v^{q-1} \text{ in } \mathcal{D}, \quad v > 0 \text{ in } \mathcal{D}, \quad \partial_\nu v = 0 \text{ on } \partial\mathcal{D}$$

where \mathcal{D} is an open bounded domain in \mathbb{R}^N , ε is a positive parameter, $q > 2$ and ν is the unit inner normal at the boundary.

Problem (1) has deserved a lot of attention in the last decades, because it is a model for different problems in applied science. It arises for instance as the shadow system associated to activator-inhibitor systems in mathematical theory of biological pattern formation such as the Gierer-Meinhardt model [21] and in the Keller-Segel model of chemotaxis [28]. A challenging feature of solutions to (1) is that they exhibit concentration phenomena as either the parameter ε approaches zero or the exponent q approaches some critical values.

1.1. The singularly perturbed problem, i.e. $\varepsilon \rightarrow 0$. In the subcritical case, i.e. $q < \frac{2N}{N-2}$ problem (1) has a least energy solution which is obtained by minimizing the Rayleigh quotient

$$(2) \quad Q(u) = \frac{\varepsilon^2 \int_{\mathcal{D}} |\nabla u|^2 + \int_{\mathcal{D}} |u|^2}{\left(\int_{\mathcal{D}} |u|^q \right)^{\frac{2}{q}}}, \quad u \in H^1(\mathcal{D}) \setminus \{0\},$$

for small ε . In a series of papers Lin, Ni and Takagi [32, 38, 39] proved that if ε is small enough the least energy solution has a unique local maximum point ξ_ε which is located on the boundary $\partial\mathcal{D}$ and approaches as ε goes to zero the maximum of the mean curvature of the boundary, i.e. $H(\xi_\varepsilon) \rightarrow \max_{\xi \in \partial\mathcal{D}} H(\xi)$ as $\varepsilon \rightarrow 0$. Here and in the following H denotes mean curvature of $\partial\mathcal{D}$. Moreover, this solution decays exponentially far away from the maximum point which implies indeed the presence of a very sharp, bounded spike for the solution around ξ_ε . Higher energy solutions with similar qualitative behavior have been found by several authors: solutions with boundary peaks in [10, 11, 22, 27, 30, 51], with interior peaks in [6, 9, 19, 20, 25, 29, 50, 53] or with both boundary and interior peaks in [26]. In

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particular, we quote the result in [51], where the author proved that such a spike solution exists around any non-degenerate critical point of the mean curvature.

Phenomena of this type occurs as well in the critical case $q = \frac{2N}{N-2}$, however several important differences are present. For instance, since compactness of the embedding of $H^1(\mathcal{D})$ into $L^q(\mathcal{D})$ is lost, existence of minimizers of $Q(u)$ becomes non-obvious. (and in general not true for large ε as established in [31]). It is the case however, as shown in [1, 47], that such a minimizer does exist if ε is sufficiently small. The profile and asymptotic behavior of this least energy solution has been analyzed in [4, 5, 37, 43]. Again the least energy solution has only one local maximum point located around a point of maximum mean curvature of $\partial\Omega$. Unlike the subcritical case, the L^∞ -norm of the least energy solution is unbounded as ε goes to zero. Construction of solutions with this type of blowing-up behavior around one or more critical points of the mean curvature has been achieved for instance in [2, 3, 18, 23, 24, 33, 41, 43, 46, 47, 48, 49, 52]. An important difference with the subcritical case is that now mean curvature is required to be positive at these critical points. Indeed, non-negativity of curvature is actually necessary for existence [24, 43]. Moreover, in contrast with the subcritical situation, there are no solutions which blow-up only in interior points, i.e. at least one blow-up point has to lie on the boundary as established in [7, 41].

1.2. The almost critical problem, i.e. $q \rightarrow \frac{2N}{N-2}$ if $N \geq 3$ or $q \rightarrow +\infty$ if $N = 2$. The first result in this direction is obtained in [8], where the authors proved that if $N \geq 4$ and if the exponent q approaches the critical exponent *from below*, i.e. $q = \frac{2N}{N-2} + \epsilon$ with ϵ small and negative, then there exists a solution blowing-up at points located on the boundary near critical points of the mean curvature with *negative value*.

Recently, Rey and Wei in [45] and del Pino, Musso and Pistoia [17] proved that if $N \geq 4$ and if the exponent q approaches the critical exponent *from above*, i.e. $q = \frac{2N}{N-2} + \epsilon$ with ϵ small and positive, then there exists a solution blowing-up at points located on the boundary near critical points of the mean curvature with *positive value*. The case $N = 3$ is more involved and it has been treated by Rey and Wei in [44].

If $N = 2$ the exponent q is allowed to go to $+\infty$ and Musso and Wei in [35] constructed solutions concentrating in interior and boundary points, whose location is determined by a suitable combination of Green's functions.

1.3. Concentration along higher dimensional sets. It is natural to look for solutions to problem (1) that exhibit concentration phenomena not just at points but on higher dimensional subsets of $\overline{\mathcal{D}}$ as suggested by Ni in [36].

It is useful to introduce the *h -th critical exponent*. Given an integer h we define

$$q_h^* := \frac{2(N-h)}{N-h-2} \text{ if } 0 \leq h \leq N-3 \quad \text{and} \quad q_{N-2}^* := +\infty \text{ if } h = N-2.$$

q_h^* is nothing but the critical Sobolev exponent of the embedding $H^1(\Omega) \hookrightarrow L^{q_h^*}(\Omega)$ where Ω is a smooth bounded domain in R^{N-h} . We also note that $q_0^* = \frac{2N}{N-2}$.

In particular, given Γ a h -dimensional submanifold of the boundary and assuming that $1 \leq h \leq N-3$ and $q \leq q_h^*$, the question is whether there exists a solution to (1) which concentrates along Γ as $\varepsilon \rightarrow 0$.

In [13, 14, 15, 16] the authors have established the existence of such a solution (for a suitable sequence of parameters $\varepsilon_i \rightarrow 0$) in the h -subcritical case, i.e. $q < q_h^*$, when either Γ is the whole boundary or Γ is an embedded closed minimal

submanifold of $\partial\mathcal{D}$ which is in addition nondegenerate in the sense that its Jacobi operator is nonsingular.

Recently, in [12] the result has been extended to the h -critical case, i.e. $q = q_h^*$. The authors assume that Γ is an embedded closed minimal submanifold of $\partial\mathcal{D}$ whose dimension is $h \leq N - 7$ (in particular, $N \geq 8$) which is nondegenerate, and a certain weighted average of sectional curvatures of the boundary $\partial\mathcal{D}$ is positive along Γ . Then they prove the existence of a solution to problem (1) for a suitable sequence of parameters $\varepsilon_i \rightarrow 0$ which blows-up along Γ .

As far as we know, there are not any results about existence of this kind of solutions to problem (1) when the parameter ε is fixed (say $\varepsilon = 1$) and the exponent q is allowed to approach the higher critical exponents q_h^* . In particular, a natural question arises

(Q) *for any integer $h = 1, \dots, N - 3$, if q approaches q_h^* , does problem (3) have a positive solution which blows-up along a suitable h -dimensional minimal submanifold of the boundary $\partial\mathcal{D}$?*

In the present paper, we give a positive answer when the domain \mathcal{D} has some rotational symmetry.

1.4. Our result. Let $n \geq 1$ and $n \geq m \geq 1$ be fixed integers. Let Ω be a smooth open bounded domain in \mathbb{R}^n such that

$$\overline{\Omega} \subset \{(x_1, \dots, x_m, x') \in \mathbb{R}^m \times \mathbb{R}^{n-m} : x_i > 0, i = 1, \dots, m\}.$$

Let $M = M_1 + \dots + M_m$, $M_i \geq 2$, and set

$$\mathcal{D} := \{(y_1, \dots, y_m, x') \in \mathbb{R}^{M_1} \times \dots \times \mathbb{R}^{M_m} \times \mathbb{R}^{n-m} : (|y_1|, \dots, |y_m|, x') \in \Omega\}.$$

Then \mathcal{D} is a smooth bounded domain in \mathbb{R}^N , $N := M + n - m$. Set $h := M - m$, so that $N - h = n$.

The solutions we are looking for are \mathcal{G} -invariant for the action of the group $\mathcal{G} := \mathcal{O}(M_1) \times \dots \times \mathcal{O}(M_m)$ on \mathbb{R}^N given by

$$(g_1, \dots, g_m)(y_1, \dots, y_m, x') := (g_1 y_1, \dots, g_m y_m, x').$$

Here $\mathcal{O}(M_i)$ denotes the group of linear isometries of \mathbb{R}^{M_i} .

A simple calculation shows that a function v of the form $v(y_1, \dots, y_m, x') = u(|y_1|, \dots, |y_m|, x')$ solves problem

$$(3) \quad -\Delta v + v = v^{q-1} \text{ in } \mathcal{D}, \quad v > 0 \text{ in } \mathcal{D}, \quad \partial_\nu v = 0 \text{ on } \partial\mathcal{D}$$

if u solves

$$(4) \quad -\Delta u + \sum_{i=1}^m \frac{M_i - 1}{x_i} \frac{\partial u}{\partial x_i} + u = u^{q-1} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad \partial_\nu u = 0 \text{ on } \partial\Omega.$$

Here $q = 2_h^* = \frac{2n}{n-2} + \epsilon$ where ϵ is a small *positive* or *negative* parameter. Thus, we are lead to study the more general anisotropic problem

$$(5) \quad -\operatorname{div}(a(x)\nabla u) + a(x)u = a(x)u^{q-1} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad \partial_\nu u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded smooth domain in \mathbb{R}^n , $a \in C^2(\overline{\Omega})$ and $\min_{x \in \overline{\Omega}} a(x) > 0$. Note that if

$$(6) \quad a(x_1, \dots, x_m) := x_1^{M_1-1} \dots x_m^{M_m-1},$$

problem (5) reduces to problem (4).

Our goal is to construct solutions to problem (5) which blow-up at a suitable critical point ξ_0 of a constrained on the boundary $\partial\Omega$ as ϵ goes to 0. It corresponds

to construct solutions to problem (3) which blow-up along the \mathcal{G} -orbit $\Xi(\xi_0)$ of ξ_0 lying on the boundary $\partial\mathcal{D}$ as ϵ goes to 0. Here

$$(7) \quad \Xi(\xi_0) := \{(y_1, \dots, y_m, x') \in \partial D : (|y_1|, \dots, |y_m|, x') = \xi_0 \in \partial\Omega\}$$

is a h -dimensional minimal submanifold of the boundary of \mathcal{D} diffeomorphic to $\mathbb{S}^{M_1-1} \times \dots \times \mathbb{S}^{M_m-1}$ (recall that $M - m = h$), where \mathbb{S}^{M_i-1} is the unit sphere in \mathbb{R}^{M_i} .

Set

$$(8) \quad \mathcal{H}_a(\xi) := \frac{2}{n-1} \frac{\partial_\nu a(\xi)}{a(\xi)} - H(\xi).$$

Our main result is the following.

Theorem 1. *Let $N - h \geq 5$. Assume ξ_0 is a C^1 -stable critical point of a constrained on the boundary.*

- (i) *If $\mathcal{H}_a(\xi_0) < 0$ there exists $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$ problem (3) has a positive solution v_ϵ which blows-up along $\Xi(\xi_0)$ (see (7)) as $\epsilon \rightarrow 0$.*
- (ii) *If $\mathcal{H}_a(\xi_0) > 0$ there exists $\epsilon_0 > 0$ such that for each $\epsilon \in (-\epsilon_0, 0)$ problem (3) has a positive solution v_ϵ which blows-up along $\Xi(\xi_0)$ (see (7)) as $\epsilon \rightarrow 0$.*

We recall that any strict local maximum point or strict local minimum point or non-degenerate critical point is a C^1 -stable critical point of a (see for instance [30]).

Remark 1. *We point out that the sign of $\mathcal{H}_a(\xi_0)$ is nothing but the sign of a weighted average of sectional curvatures of the submanifold $\Xi(\xi_0)$ (see (7)). Indeed a simple calculation shows that if a is given as in (6) then*

$$\frac{\partial_\nu a(\xi)}{a(\xi)} = \frac{M_1 - 1}{x_1} \nu_1 + \dots + \frac{M_m - 1}{x_m} \nu_m,$$

where ν_1, \dots, ν_m are the first m -components of the unit inner normal ν at the boundary point ξ . A similar weighted average of sectional curvatures was introduced in (1.13) of [12].

According to the previous discussion, Theorem 1 is an immediate consequence of the following result.

Theorem 2. *Let $n \geq 5$. Assume ξ_0 is a C^1 -stable critical point of a constrained on the boundary.*

- (i) *If $\mathcal{H}_a(\xi_0) < 0$ there exists $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$, problem (5) has a solution u_ϵ which blows-up at ξ_0 as $\epsilon \rightarrow 0$.*
- (ii) *If $\mathcal{H}_a(\xi_0) > 0$ there exists $\epsilon_0 > 0$ such that, for each $\epsilon \in (-\epsilon_0, 0)$, problem (5) has a solution u_ϵ which blows-up at ξ_0 as $\epsilon \rightarrow 0$.*

It is useful to give a simple example.

Example 1. *Let Ω be a strict convex domain in \mathbb{R}^{N-1} , so that $H(\xi) > 0$ for any $\xi \in \partial\Omega$. Let D the torus-like domain*

$$\mathcal{D} := \{(y_1, x') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : (|y_1|, x') \in \Omega\}.$$

In particular $h = 1$, $n = N - 1$ ($m = 1$ and $M = 2$) and $a(x) = x_1$. In this case the exponent q approaches $q_1^ = \frac{2(N-1)}{N-3}$. It is clear that a constrained on $\partial\Omega$ has two C^1 -stable critical points: a strict minimum point ξ_m and a strict maximum point ξ_M , i.e.*

$$a(\xi_m) := \min_{\xi \in \partial\Omega} \xi_1 \quad \text{and} \quad a(\xi_M) := \max_{\xi \in \partial\Omega} \xi_1.$$

It is immediate to check that

$$\mathcal{H}_a(\xi_m) = c_N \left(\frac{2}{N-2} \frac{1}{a(\xi_m)} - H(\xi_1) \right) \text{ and } \mathcal{H}_a(\xi_M) = c_N \left(-\frac{2}{N-2} \frac{1}{a(\xi_M)} - H(\xi_2) \right).$$

In particular, $\mathcal{H}_a(\xi_M) < 0$. On the other hand, $\mathcal{H}_a(\xi_m) < 0$ if $a(\xi_m)$ is large enough while $\mathcal{H}_a(\xi_m) > 0$ if $a(\xi_m)$ is small enough.

It is interesting to note that in this case problem (1) has always a solution when the exponent q approaches 2_1^* from above, which concentrate along the geodesic $\Xi(\xi_M)$. On the other hand, if the exponent q approaches 2_1^* from below existence of solutions to problem (1) depends on the size of the hole of \mathcal{D} , i.e. if $a(\xi_m)$ is small enough then there exists a solution which concentrate along the geodesic $\Xi(\xi_m)$.

The proof of our results relies on a very well known Ljapunov-Schmidt reduction. We omit many details of the finite dimensional reduction, because they can be found, up to some minor modifications, in the literature. In Section 2 we sketch the main steps of the proof and we prove Theorem 2. The proofs which can not be immediately deduced from known results are given in the Appendix.

2. SCHEME OF THE PROOF

2.1. Setting of the problem. We introduce the Hilbert space $H^1(\Omega)$ equipped with the inner product and the corresponding norm

$$(u, v) := \int_{\Omega} a(x) \nabla u \nabla v \, dx \text{ and } \|u\| := \left(\int_{\Omega} a(x) |\nabla u|^2 \, dx \right)^{1/2}.$$

We also introduce the Banach space $L^r(\Omega)$, $r \in [1, \infty)$, equipped with the norm

$$|u|_r := \left(\int_{\Omega} a(x) |u|^r \, dx \right)^{1/r}.$$

Let $i^* : L^{\frac{2n}{n-2}}(\Omega) \rightarrow H^1(\Omega)$ be the adjoint operator of the embedding $i : H^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$, i.e. $i^*(f) = u$ if and only if

$$-\operatorname{div}(a(x) \nabla u) + a(x)u = a(x)f \quad \text{in } \Omega, \quad \partial_{\nu} u = 0 \quad \text{on } \partial\Omega.$$

It is clear that there exists a positive constant c such that

$$(9) \quad \|i^*(f)\| \leq c|f|_{\frac{2n}{n-2}} \quad \forall f \in L^{\frac{2n}{n-2}}(\Omega).$$

To study problem (5) in the supercritical case, it is useful to recall this regularity result (see for example [34]). inequality

$$(10) \quad |i^*(f)|_s \leq C|f|_{\frac{ns}{n+2s}} \quad \text{for } f \in L^{\frac{ns}{n+2s}}$$

where $s \geq \frac{2n}{n-2}$ so that $\frac{ns}{n+2s} \geq \frac{2n}{n-2}$.

Next, we consider the Banach space $H_{\epsilon} = H^1(\Omega) \cap L^{s_{\epsilon}}(\Omega)$ with the norm

$$\|u\|_{H_{\epsilon}} = \|u\| + |u|_{s_{\epsilon}}$$

where we set $s_{\epsilon} = \frac{2n}{n-2} + \epsilon$ if $\epsilon > 0$ and $s_{\epsilon} = \frac{2n}{n-2}$ if $\epsilon \leq 0$. We remark that if $\epsilon \leq 0$ the space H_{ϵ} is nothing but the space $H^1(\Omega)$ with the norm $\|\cdot\|$. Finally, also using the maximum principle, problem (5) is equivalent to the problem

$$(11) \quad u = i^*(f_{\epsilon}(u)) \quad u \in H_{\epsilon}$$

where $f_{\epsilon}(u) = (u^+)^{p+\epsilon}$, $u^+ = \max\{u, 0\}$ and $p := \frac{n+2}{n-2}$.

2.2. The approximated solution. Let us introduce the *standard bubbles*

$$U_{\delta,\xi} := \alpha_n \frac{\delta^{\frac{n-2}{2}}}{(\delta^2 + |x - \xi|^2)^{\frac{n-2}{2}}}, \quad \delta > 0, \quad \xi \in \mathbb{R}^n,$$

with $\alpha_n := [n(n-2)]^{\frac{n-2}{4}}$, which are the positive solutions to the limit problem

$$-\Delta u = u^p, \quad u \in H^1(\mathbb{R}^n).$$

It is useful to recall that the set of solutions to the linearized problem

$$-\Delta Z = pU_{\delta,\xi}^{p-1} Z \text{ in } \mathbb{R}^n.$$

is spanned by the functions

$$Z_{\delta,\xi}^0(x) := \frac{\partial U_{\delta,\xi}}{\partial \delta} = \alpha_n \frac{n-2}{2} \delta^{\frac{n-4}{2}} \frac{|x - \xi|^2 - \delta^2}{(\delta^2 + |x - \xi|^2)^{n/2}}$$

and, for each $j = 1, \dots, n$,

$$Z_{\delta,\xi}^j(x) := \frac{\partial U_{\delta,\xi}}{\partial \xi_j} = \alpha_n (n-2) \delta^{\frac{n-2}{2}} \frac{x_j - \xi_j}{(\delta^2 + |x - \xi|^2)^{n/2}}.$$

Let $PU_{\delta,\xi}$ denote the solution of the problem onto $H^1(\Omega)$, i.e.

$$-\Delta PU_{\delta,\xi} + PU_{\delta,\xi} = U_{\delta,\xi}^p \text{ in } \Omega, \quad \partial_\nu PU_{\delta,\xi} = 0 \text{ on } \partial\Omega.$$

We look for a solution to problem (11) as

$$u_\epsilon = PU_{\delta,\xi} + \phi_\epsilon,$$

where the concentration point and the concentration parameter satisfy

$$\xi \in \partial\Omega \quad \text{and} \quad \delta = |\epsilon|d \quad \text{for some } d > 0.$$

The rest term ϕ_ϵ belongs to a suitable space defined as follows. Let us introduce the spaces

$$K_{d,\xi} := \text{span}\{Z_{\delta,\xi}^j : j = 0, 1, \dots, n\}$$

and

$$K_{d,\xi}^\perp := \left\{ \phi \in H^1(\Omega) : (\phi, Z_{\delta,\xi}^j) = 0, \quad j = 0, 1, \dots, n \right\},$$

and the projection operators

$$\Pi_{d,\xi}(u) := \sum_{j=0}^n (u, Z_{\delta,\xi}^j) Z_{\delta,\xi}^j \quad \text{and} \quad \Pi_{d,\xi}^\perp(u) := u - \Pi_{d,\xi}(u).$$

As usual, our approach to solve problem (11) will be to find a $(d, \xi) \in \mathbb{R} \times \partial\Omega$ and a function $\phi \in K_{d,\xi}^\perp$ such that

$$(12) \quad \Pi_{d,\xi}^\perp \{PU_{\delta,\xi} + \phi - i^*[f_\epsilon(PU_{\delta,\xi} + \phi)]\} = 0$$

and

$$(13) \quad \Pi_{d,\xi} \{PU_{\delta,\xi} + \phi - i^*[f_\epsilon(PU_{\delta,\xi} + \phi)]\} = 0.$$

2.3. Reduction to a finite dimensional problem: solving equation (12). First, we find for any $(d, \xi) \in \mathbb{R} \times \partial\Omega$ and small ϵ a function $\phi \in K_{d,\xi}^\perp$ such that (12) holds. To this aim we define a linear operator $L_{d,\xi} : K_{d,\xi}^\perp \rightarrow K_{d,\xi}^\perp$ by

$$(14) \quad L_{d,\xi}\phi := \phi - \Pi_{d,\xi}^\perp [f'_\epsilon(PU_{\delta,\xi})\phi].$$

We will prove the following.

Proposition 1. *For any compact subset C of $\mathbb{R} \times \partial\Omega$ there exist $\epsilon_0 > 0$ and $c > 0$ such that for each $\epsilon \in (-\epsilon_0, \epsilon_0)$ and $(d, \xi) \in C$ the operator $L_{d,\xi}$ is invertible and*

$$\|L_{d,\xi}\phi\|_{H_\epsilon} \geq c \|\phi\|_{H_\epsilon} \quad \forall \phi \in K_{d,\xi}^\perp.$$

Proof. We argue as in Lemma 3.1 of [34]. \square

Now, we are in position to solve equation (12).

Proposition 2. *For any compact subset C of $\mathbb{R} \times \partial\Omega$ there exist $\epsilon_0 > 0$ and $c > 0$ such that for each $\epsilon \in (-\epsilon_0, \epsilon_0)$ and $(d, \xi) \in C$ there exists a unique $\phi_{d,\xi}^\epsilon \in K_{d,\xi}^\perp$ which solves (12) and satisfies*

$$\|\phi_{d,\xi}^\epsilon\|_{H_\epsilon} \leq c|\epsilon| |\ln \epsilon|.$$

Moreover, the map $(d, \xi) \rightarrow \phi_{d,\xi}^\epsilon$ is a C^1 -map which satisfies

$$\|D_{(d,\xi)}\phi_{d,\xi}^\epsilon\|_{H_\epsilon} \leq c|\epsilon| |\ln \epsilon|.$$

Proof. The proof is postponed to Appendix. \square

2.4. The reduced problem: solving equation (13). We now introduce the energy functional $J_\epsilon : H \rightarrow \mathbb{R}$ defined by

$$J_\epsilon(u) := \frac{1}{2} \int_{\Omega} a(x) (|\nabla u|^2 + u^2(x)) dx - \frac{1}{p+1+\epsilon} \int_{\Omega} a(x) (u^+)^{p+1+\epsilon} dx,$$

whose critical points are the solutions to problem (11). Let us define the reduced energy functional $\tilde{J}_\epsilon : \mathbb{R} \times \partial\Omega \rightarrow \mathbb{R}$ by

$$\tilde{J}_\epsilon(d, \xi) := J_\epsilon(PU_{\delta,\xi} + \phi_{d,\xi}^\epsilon)$$

Next, we prove that the critical points of \tilde{J}_ϵ are the solutions to problem (13).

Proposition 3. *The function $PU_{\delta,\xi} + \phi_{d,\xi}^\epsilon$ is a critical point of the functional J_ϵ if and only if the point (d, ξ) is a critical point of the function \tilde{J}_ϵ .*

Proof. We argue as in Proposition 2.2 of [34]. \square

The problem is thus reduced to finding critical points of \tilde{J}_ϵ and so it is necessary to compute the asymptotic expansion of \tilde{J}_ϵ .

Proposition 4. *It holds true that*

$$\tilde{J}_\epsilon(d, \xi) = a(\xi) [c_1 + c_2\epsilon \log |\epsilon| + c_3\epsilon + c_4\mathcal{H}_a(\xi)|\epsilon|d + c_5\epsilon \ln d + o(\epsilon)],$$

C^1 -uniformly on compact sets of $\mathbb{R} \times \partial\Omega$. Here $\mathcal{H}_a(\xi)$ is defined in (8), c_i are constants and in particular, c_4 and c_5 are positive.

Proof. The proof is postponed to Appendix. \square

2.5. Proof of Theorem 2.

We apply Proposition 3. Then it is enough to prove that the reduced energy \tilde{J}_ϵ has a critical point if ϵ is small enough. By Proposition 4 we deduce that

$$\nabla_\xi \tilde{J}_\epsilon(d, \xi) = c_1 \nabla_\xi a(\xi) + o(1).$$

Moreover

$$\nabla_d \tilde{J}_\epsilon(d, \xi) = \left[c_4 \mathcal{H}_a(\xi) + c_5 \frac{1}{d} + o(1) \right] \epsilon \text{ if } \epsilon > 0,$$

and

$$\nabla_d \tilde{J}_\epsilon(d, \xi) = \left[-c_4 \mathcal{H}_a(\xi) + c_5 \frac{1}{d} + o(1) \right] \epsilon \text{ if } \epsilon < 0.$$

Let ξ_0 be a C^1 -stable critical point of a constrained on the boundary. By Brouwer degree theory it follows that if either $d_0 := -\frac{c_5}{c_4} \frac{1}{\mathcal{H}_a(\xi_0)}$ if $\epsilon > 0$ or $d_0 := \frac{c_5}{c_4} \frac{1}{\mathcal{H}_a(\xi_0)}$ if $\epsilon < 0$, if ϵ is small enough there exists $(d_\epsilon, \xi_\epsilon)$ such that $\nabla_{(d, \xi)} \tilde{J}_\epsilon(d_\epsilon, \xi_\epsilon) = 0$, $d_\epsilon \rightarrow d_0$ and $\xi_\epsilon \rightarrow \xi_0$ as $\epsilon \rightarrow 0$. That proves our claim.

APPENDIX A.

A.1. A local parametrization of the boundary. Take $\xi \in \partial\Omega$. Without loss of generality we may take $\xi = 0$ and the unit inward normal of $\partial\Omega$ at ξ directed along the x_n -axis. Denote $x' = (x_1, x_2, \dots, x_{n-1})$, $B'(0, r) = \{x' \in \mathbb{R}^{n-1} : |x'| < r\}$, and $\Omega_1 := \Omega \cap B(\xi, r)$, where $B(\xi, r) = \{x \in \mathbb{R}^n : |x - \xi| < r\}$. Since $\partial\Omega$ is smooth, we can find a constant $r > 0$ such that $\partial\Omega \cap B(\xi, r)$ can be represented by the graph of a smooth function $\rho_\xi : B'(0, r) \rightarrow \mathbb{R}$, where $\rho_\xi(0) = 0$ and $\nabla \rho_\xi(0) = 0$, and

$$\Omega \cap B(\xi, r) = \{(x', x_n) \in B(\xi, r) : x_n > \rho_\xi(x')\}.$$

Moreover, we may write

$$\rho_\xi(x') = \frac{1}{2} \sum_{i=1}^{n-1} k_i x_i^2 + O(|x|^3)$$

where $k_i, i = 1, 2, \dots, n-1$, are the principle curvatures at ξ . The mean curvature is defined by

$$H(\xi) = \frac{1}{n-1} \sum_{i=1}^{n-1} k_i$$

A.2. The expansion of the ansatz. From Lemma A.1 of [45] we have that

$$(15) \quad PU_{\delta, \xi}(x) = U_{\delta, \xi}(x) - \delta^{\frac{4-n}{2}} \varphi_0 \left(\frac{x - \xi}{\delta} \right) + O \left(\delta^{\frac{6-n}{2}} \right)$$

where φ_0 solves

$$(16) \quad \begin{cases} \Delta \varphi_0 = 0 & \text{in } \mathbb{R}_+^n \\ \frac{\partial \varphi_0}{\partial x_n} = \alpha_n \frac{n-2}{2} \sum_{i=1}^{n-1} \frac{k_i x_i^2}{(1 + |x|^2)^{n/2}} & \text{on } \partial \mathbb{R}_+^n \\ \varphi_0 \rightarrow 0 & \text{as } |x| \rightarrow \infty \end{cases}$$

Using Green's expression φ_0 can be expressed as

$$\varphi_0(x) = \alpha_n \frac{1}{\omega_{n-1}} \sum_{i=1}^{n-1} k_i \int_{\mathbb{R}^{n-1}} \frac{y_i^2}{(1 + |y'|^2)^{n/2}} \frac{1}{|x - y'|^{N-2}} dy'$$

where ω_{n-1} is the surface measure of the unit sphere in \mathbb{R}^n . Moreover

$$(17) \quad |\varphi_0(x)| \leq \frac{C}{(1 + |x|)^{n-3}}.$$

A.3. Proof of Proposition 2. As it is usual equation (12) turns out to be equivalent to

$$L_{d,\xi}(\phi) = N_{d,\xi}(\phi) + R_{d,\xi},$$

where the linear operator $L_{d,\xi}$ is defined in (14), the error-term is

$$(18) \quad R_{d,\xi} := \Pi_{d,\xi}^\perp \{i^* [f_\epsilon (PU_{\delta,\xi})] - PU_{\delta,\xi}\}$$

and the higher order term is

$$N_{d,\xi}(\phi) := \Pi_{d,\xi}^\perp \{i^* [f_\epsilon (PU_{\delta,\xi} + \phi) - f_\epsilon (PU_{\delta,\xi}) - f'_\epsilon (PU_{\delta,\xi}) \phi]\}$$

By Proposition 1, using Lemma 2 and the usual contraction mapping argument, the claim follows (see, for instance, Proposition 2.1 of [34]).

We recall the following useful lemma.

Lemma 1. *For any $a > 0$ and $b \in \mathbb{R}$ we have*

$$||a + b|^\beta - a^\beta| \leq \begin{cases} c(\beta) \min\{|b|^\beta, a^{\beta-1}|b|\} & \text{if } 0 < \beta < 1 \\ c(\beta) (|b|^\beta + a^{\beta-1}|b|) & \text{if } \beta > 1 \end{cases}$$

and

$$||a + b|^\beta (a + b) - a^{\beta+1} - (1 + \beta)a^\beta b| \leq \begin{cases} c(\beta) \min\{|b|^{\beta+1}, a^{\beta-1}b^2\} & \text{if } 0 < \beta < 1 \\ c(\beta) \max\{|b|^{\beta+1}, a^{\beta-1}b^2\} & \text{if } \beta > 1 \end{cases}$$

Let us estimate the error term (18).

Lemma 2. *It holds*

$$\|R_{d,\xi}\|_{H_\epsilon} = O(|\epsilon| \ln |\epsilon|).$$

Proof. By the definition of i^* , we immediately get that

$$PU_{\delta,\xi} = i^* \left[f_0(U_{\delta,\xi}) - \frac{\nabla a(x)}{a(x)} \nabla PU_{\delta,\xi} \right].$$

Then by (9) and (10) we get

$$(19) \quad \begin{aligned} \|R_{d,\xi}\|_{H_\epsilon} &= O \left(|f_\epsilon(PU_{\delta,\xi}) - f_0(PU_{\delta,\xi})|_{\frac{2n}{n+2}} + |f_\epsilon(PU_{\delta,\xi}) - f_0(PU_{\delta,\xi})|_{\frac{ns_\epsilon}{n+2s_\epsilon}} \right) \\ &\quad + O \left(|f_0(PU_{\delta,\xi}) - f_0(U_{\delta,\xi})|_{\frac{2n}{n+2}} + |f_0(PU_{\delta,\xi}) - f_0(U_{\delta,\xi})|_{\frac{ns_\epsilon}{n+2s_\epsilon}} \right) \\ &\quad + O \left(\left| \frac{\nabla a(x)}{a(x)} \nabla PU_{\delta,\xi} \right|_{\frac{2n}{n+2}} + \left| \frac{\nabla a(x)}{a(x)} \nabla PU_{\delta,\xi} \right|_{\frac{ns_\epsilon}{n+2s_\epsilon}} \right) \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Since $\frac{ns_\epsilon}{n+2s_\epsilon} = \frac{2n}{n+2} + O(\epsilon)$, we only compute the $L^{\frac{2n}{n+2}}$ -norms. Using the same arguments of Proposition 2 of [40] we can estimate I_1 as

$$(20) \quad |f_\epsilon(PU_{\delta,\xi}) - f_0(PU_{\delta,\xi})|_{\frac{2n}{n+2}} = O(|\epsilon| \ln |\epsilon|).$$

Let us estimate I_2 . By Lemma 1, (15) and (17) we deduce that

$$(21) \quad \begin{aligned} |f_0(PU_{\delta,\xi}) - f_0(U_{\delta,\xi})|_{\frac{2n}{n+2}} &= O \left(|PU_{\delta,\xi} - U_{\delta,\xi}|_{\frac{2n}{n+2}}^p \right) + O \left(U_{\delta,\xi}^{p-1} (PU_{\delta,\xi} - U_{\delta,\xi}) \right)_{\frac{2n}{n+2}} \\ &= O(\delta) \text{ (because } n \geq 5). \end{aligned}$$

Finally, we estimate I_3 as

$$(22) \quad \left| \frac{\nabla a(x)}{a(x)} \nabla PU_{\delta,\xi} \right|_{\frac{2n}{n+2}} = O(\delta) \text{ (because } n \geq 5).$$

By (19), (20), (21) and (22) the claim follows. \square

A.4. Proof of Proposition 4. First of all it is quite standard to prove that (see, for instance, Proposition 2.2 of [34])

$$(23) \quad J_\epsilon(PU_{\delta,\xi} + \phi_{d,\xi}^\epsilon) = J_\epsilon(PU_{\delta,\xi}) + o(\epsilon).$$

Therefore, we only have to estimate $J_\epsilon(PU_{\delta,\xi})$. We remark that

$$(24) \quad \begin{aligned} J_\epsilon(PU_{\delta,\xi}) &= a(\xi) \left[\frac{1}{2} \int_{\Omega} (|\nabla u|^2 + u^2) dx - \frac{1}{p+1+\epsilon} \int_{\Omega} (u^+)^{p+1+\epsilon} dx \right] \\ &\quad + \int_{\Omega} [a(x) - a(\xi)] \left[\frac{1}{2} (|\nabla u|^2 + u^2) - \frac{1}{p+1+\epsilon} (u^+)^{p+1+\epsilon} \right] dx \\ &:= I_1 + I_2 \end{aligned}$$

I_1 was estimated in Proposition A.1 of [45] (note that in our case the expansion in [45] also contain α_n^2) as

$$(25) \quad I_1 = a(\xi) [c_1 + c_2\epsilon \ln |\epsilon| + c_3\epsilon + c_5\epsilon \ln d - c_6d|\epsilon|H(\xi) + o(\epsilon)]$$

where c_i are constants. In particular c_5 is positive and

$$c_6 := \frac{(n-2)^2}{n-3} \alpha_n^2 \int_{\mathbb{R}^{n-1}} \frac{|y'|^2}{(1+|y'|^2)^n} dy'$$

Let us estimate I_2 taking into account that

$$a(x) = a(\xi) + \nabla a(x)(x - \xi) + O(|x - \xi|^2).$$

Then we have

$$(26) \quad \begin{aligned} I_2 &= \int_{\Omega} [\nabla a(x)(x - \xi) + O(|x - \xi|^2)] \left[\frac{1}{2} (|\nabla u|^2 + u^2) - \frac{1}{p+1+\epsilon} (u^+)^{p+1+\epsilon} \right] dx \\ &\quad (\text{scaling } x - \xi = \delta y) \\ &= \delta \left[\nabla a(\xi) \int_{\mathbb{R}_+^n} y \left[\frac{1}{2} |\nabla U|^2 - \frac{1}{p+1} U^{p+1} \right] dy + o(1) \right] \\ &= \delta \left[\partial_\nu a(\xi) \int_{\mathbb{R}_+^n} y_n \left[\frac{(n-2)^2}{2} \alpha_n^2 \frac{|y|^2}{(1+|y|^2)^n} - \frac{1}{p+1} \alpha_n^{p+1} \frac{1}{(1+|y|^2)^n} \right] dy + o(1) \right] \\ &= c_7 d |\epsilon| \partial_\nu a(\xi) + o(\epsilon), \end{aligned}$$

where

$$c_7 := \frac{(n-2)^2}{2} \alpha_n^2 \int_{\mathbb{R}_+^n} y_n \frac{|y|^2 - 1}{(1+|y|^2)^n} dy.$$

We need to compare the two constants c_6 and c_7 . First of all, we have (since $n \geq 5$)

$$\int_{\mathbb{R}_+^n} y_n \frac{|y|^2 - 1}{(1+|y|^2)^n} dy = \frac{1}{2(n-1)(n-2)} \int_{\mathbb{R}^{n-1}} \frac{1}{(1+|y'|^2)^{n-2}} dy' + \frac{1}{2(n-1)} \int_{\mathbb{R}^{n-1}} \frac{|y'|^2 - 1}{(1+|y'|^2)^{n-1}} dy'$$

For any positive real numbers p and q such that $p - q > 1$, we let

$$I_p^q = \int_0^{+\infty} \frac{r^q}{(1+r)^p} dr = 2 \int_0^{+\infty} \frac{s^{2q+1}}{(1+s^2)^p} ds = \frac{\Gamma(q+1)\Gamma(p-q-1)}{\Gamma(p)},$$

where Γ is the Gamma Eulero function. In particular, we have

$$I_{p+1}^q = \frac{p-q-1}{p} I_p^q \quad \text{and} \quad I_{p+1}^{q+1} = \frac{q+1}{p-q-1} I_{p+1}^q.$$

Then we can write

$$c_6 = \frac{(n-2)^2}{2(n-3)} \alpha_n^2 \omega_{n-2} I_n^{\frac{n-1}{2}}.$$

where ω_{n-2} is the measure of the $(n-2)$ -dimensional unit sphere. An easy computation leads to

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} \frac{1}{(1+|y'|^2)^{n-2}} dy' &= \frac{1}{2} \omega_{n-2} I_{n-2}^{\frac{n-3}{2}} = \frac{2(n-2)}{n-3} \omega_{n-2} I_n^{\frac{n-1}{2}}, \\ \int_{\mathbb{R}^{n-1}} \frac{|y'|^2}{(1+|y'|^2)^{n-1}} dy' &= \frac{1}{2} \omega_{n-2} I_{n-1}^{\frac{n-1}{2}} = \frac{n-1}{n-3} \omega_{n-2} I_n^{\frac{n-1}{2}} \end{aligned}$$

and

$$\int_{\mathbb{R}^{n-1}} \frac{1}{(1+|y'|^2)^{n-1}} dy' = \frac{1}{2} \omega_{n-2} I_{n-1}^{\frac{n-3}{2}} = \omega_{n-2} I_n^{\frac{n-1}{2}}$$

Then

$$c_7 = \frac{(n-2)^2}{(n-1)(n-3)} \alpha_n^2 \omega_{n-2} I_n^{\frac{n-1}{2}}.$$

By collecting, the previous estimates, we have that

$$(27) \quad -c_6 d|\epsilon| a(\xi) H(\xi) + c_7 d|\epsilon| \partial_\nu a(\xi) = \frac{(n-2)^2}{2(n-3)} \alpha_n^2 \omega_{n-2} I_n^{\frac{n-1}{2}} d|\epsilon| a(\xi) \left(\frac{2}{n-1} \frac{\partial_\nu a(\xi)}{a(\xi)} - H(\xi) \right).$$

By (23), (24), (25), (26) and (27) the claim follows.

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REFERENCES

- [1] ADIMURTHI, G. MANCINI, The Neumann problem for elliptic equations with critical nonlinearity, "A tribute in honour of G. Prodi", Scuola Norm. Sup. Pisa (1991) 9–25.
- [2] ADIMURTHI, G. MANCINI, Geometry and topology of the boundary in the critical Neumann problem, J. Reine Angew. Math. 456 (1994) 1–18.
- [3] ADIMURTHI, G. MANCINI, S.L. YADAVA, The role of the mean curvature in semilinear Neumann problem involving critical exponent. Comm. Partial Differential Equations 20 (1995), no. 3-4, 591–631.
- [4] ADIMURTHI, F. PACELLA, S.L. YADAVA, Interaction between the geometry of the boundary and positive solutions of a semilinear Neumann problem with critical nonlinearity, J. Funct. Anal. 113 (1993) 318–350.
- [5] ADIMURTHI, F. PACELLA, S.L. YADAVA, Characterization of concentration points and L^∞ -estimates for solutions of a semilinear Neumann problem involving the critical Sobolev exponent, Differential Integral Equations 8 no. 1 (1995), 41–68.
- [6] P. BATES, G. FUSCO, Equilibria with many nuclei for the Cahn-Hilliard equation, J. Differential Equations 160 (2000) 283–356.
- [7] D. CAO, E.S. NOUSSAIR, S. YAN, Existence and nonexistence of interior-peaked solution for a nonlinear Neumann problem. Pacific J. Math. 200 (2001), no. 1, 19–41.
- [8] D. CAO, E.S. NOUSSAIR, The effect of geometry of the domain boundary in an elliptic Neumann problem, Adv. Differential Equations 6 no. 8 (2001), 931–958.
- [9] G. CERAMI, J. WEI, Multiplicity of multiple interior peaks solutions for some singularly perturbed Neumann problems, Intern. Math. Res. Notes 12 (1998) 601–626.
- [10] E.N. DANCER, S. YAN, Multipeak solutions for a singularly perturbed Neumann problem, Pacific J. Math. 189 (1999) 241–262.

- [11] M. DEL PINO, P. FELMER, J. WEI, On the role of mean curvature in some singularly perturbed Neumann problems. *SIAM J. Math. Anal.* 31 (1999), no. 1, 63–79.
- [12] M. DEL PINO, F. MAHMOUDI, M. MUSSO, Bubbling on boundary submanifolds for the LinNi-Takagi problem at higher critical exponents, *J. Eur. Math. Soc.* 16 (2014) 1687–1748.
- [13] F. MAHMOUDI, A. MALCHIODI, Concentration on minimal submanifolds for a singularly perturbed Neumann problem. *Adv. Math.* 209 (2007), no. 2, 460–525.
- [14] A. MALCHIODI, M. MONTENEGRO, Boundary concentration phenomena for a singularly perturbed elliptic problem. *Comm. Pure Appl. Math.* 15, 1507–1568 (2002) Zbl 1124.35305 MR 1923818
- [15] A. MALCHIODI, M. MONTENEGRO, Multidimensional boundary-layers for a singularly perturbed Neumann problem. *Duke Math. J.* 124, 105–143 (2004)
- [16] A. MALCHIODI, Concentration at curves for a singularly perturbed Neumann problem in three-dimensional domains. *Geom. Funct. Anal.* 15, 1162–1222 (2005)
- [17] M. DEL PINO, M. MUSSO, A. PISTOIA Super-critical boundary bubbling in a semilinear Neumann problem, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 22 (2005), 45–82.
- [18] M. GROSSI, A class of solutions for the Neumann problem $-\Delta u + \lambda u = u^{(N+2)/(N-2)}$, *Duke Math. J.* 79 no. 2 (1995), 309–334.
- [19] M. GROSSI, A. PISTOIA, On the effect of critical points of distance function in superlinear elliptic problems, *Adv. Differential Equations* 5 (2000) 1397–1420. MR1785679
- [20] M. GROSSI, A. PISTOIA, J. WEI, Existence of multi-peak solutions for a semilinear elliptic problem via nonsmooth critical point theory, *Calc. Var. Partial Differential Equations* 11 (2000) 143–175.
- [21] A. GIERER, H. MEINHARDT, A theory of biological pattern formation, *Kybernetik (Berlin)* 12 (1972) 30–39.
- [22] C. GUI, Multi-peak solutions for a semilinear Neumann problem, *Duke Math. J.* 84 (1996) 739–769.
- [23] C. GUI, N. GHOUSSEB, Multi-peak solutions for a semilinear Neumann problem involving the critical Sobolev exponent, *Math. Z.* 229 no. 3 (1998), 443–474.
- [24] C. GUI, C.S. LIN, Estimates for boundary-bubbling solutions to an elliptic Neumann problem, *J. Reine Angew. Math.* 546 (2002) 201–235.
- [25] C. GUI, J. WEI, Multiple interior peak solutions for some singularly perturbed Neumann problems, *J. Differential Equations* 158 (1999) 1–27.
- [26] C. GUI, J. WEI, On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems, *Canad. J. Math.* 52 (2000) 522–538.
- [27] C. GUI, J. WEI, M. WINTER, Multiple boundary peak solutions for some singularly perturbed Neumann problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 17 (2000) 47–82.
- [28] KELLER E F AND SEGEL L A. Initiation of slime mold aggregation viewed as an instability *J. Theor. Biol.* 26 399–415. 1970.
- [29] M. KOWALCZYK, Multiple spike layers in the shadow GiererMeinhardt system: existence of equilibria and quasi-invariant manifold, *Duke Math. J.* 98 (1999) 59–111.
- [30] Y.Y. LI, On a singularly perturbed equation with Neumann boundary condition, *Comm. Partial Differential Equations* 23 (1998) 487–545.
- [31] C.-S. LIN, Locating the peaks of solutions via the maximum principle, I. The Neumann problem, *Comm. Pure Appl. Math.* 54 (2001), 1065–1095.
- [32] C.S. LIN, W.N. NI, I. TAKAGI, Large amplitude stationary solutions to a chemotaxis system, *J. Differential Equations* 72 (1988) 1–27.
- [33] C.S. LIN, L. WANG, J. WEI, Bubble accumulations in an elliptic Neumann problem with critical Sobolev exponent, *Cal. Var. Partial Differential Equations* 30 (2007) 153–182.
- [34] M. MUSSO, A. PISTOIA, Tower of bubbles for almost critical problems in general domains. *J. Math. Pures Appl.* (9) 93 (2010), no. 1, 1–40
- [35] M. MUSSO AND J. WEI, Stationary solutions to a Keller-Segel chemotaxis system, *Asymp. Anal.* 49 (2006) 217–247. MR2270892
- [36] W.-M. NI, Diffusion, cross-diffusion, and their spike-layer steady states, *Notices Amer. Math. Soc.* 45 (1998) 9–18.
- [37] W.N. NI, X.B. PAN, I. TAKAGI, Singular behavior of least-energy solutions of a semi-linear Neumann problem involving critical Sobolev exponents, *Duke Math. J.* 67 (1992) 1–20.
- [38] W.N. NI, I. TAKAGI, On the shape of least-energy solutions to a semi-linear problem Neumann problem, *Comm. Pure Appl. Math.* 44 (1991) 819–851.
- [39] W.M. NI, I. TAKAGI, Locating the peaks of least-energy solutions to a semi-linear Neumann problem, *Duke Math. J.* 70 (1993) 247–281.
- [40] O. REY, The role of the Green’s function in a nonlinear elliptic problem involving the critical Sobolev exponent, *J. Funct. Anal.* 89 (1990) 1–52.

- [41] O. REY, An elliptic Neumann problem with critical nonlinearity in three dimensional domains, *Comm. Contemp. Math.* 1 (1999) 405–449.
- [42] O. REY, The question of interior blow-up points for an elliptic Neumann problem: the critical case, *J. Math. Pures Appl.* 81 (2002) 655–696. MR1968337
- [43] O. REY, Boundary effect for an elliptic Neumann problem with critical nonlinearity, *Comm. in PDE* **22** (1997), 1055–1139.
- [44] O. REY, J. WEI, Blow-up solutions for an elliptic Neumann problem with sub- or supercritical nonlinearity, I: $N=3$, *J. Funct. Anal.* 212 (2004), no. 2, 472–499.
- [45] O. REY, J. WEI, Blowing up solutions for an elliptic Neumann problem with sub- or supercritical nonlinearity, II. $N=4$, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 22 (4) (2005) 459–484.
- [46] X.J. WANG, Neumann problem of semilinear elliptic equations involving critical Sobolev exponents, *J. Differential Equations* 93 (1991) 283–310.
- [47] Z.Q. WANG, The effect of domain geometry on the number of positive solutions of Neumann problems with critical exponents, *Differential Integral Equations* 8 (1995) 1533–1554.
- [48] Z.Q. WANG, High energy and multi-peaked solutions for a nonlinear Neumann problem with critical exponent, *Proc. Roy. Soc. Edinburgh Sect. A* 125 (1995) 1003–1029.
- [49] Z.Q. WANG, Construction of multi-peaked solution for a nonlinear Neumann problem with critical exponent, *J. Nonlinear Anal.* 27 (1996) 1281–1306.
- [50] J. WEI, On the interior spike layer solutions of singularly perturbed semilinear Neumann problems, *Tohoku Math. J.* 50 (1998) 159–178.
- [51] J. WEI, On the boundary spike layer solutions to a singularly perturbed Neumann problem, *J. Differential Equations* **134** no. 1 (1997), 104–133.
- [52] J. WEI, S. YAN, Arbitrary many boundary peak solutions for an elliptic Neumann problem with critical growth. *J. Math. Pures Appl.* (9) 88 (2007), no. 4, 350–378
- [53] S. YAN, On the number of interior multipeak solutions for singularly perturbed Neumann problems, *Topol. Methods Nonlinear Anal.* 12 (1998) 61–78.

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